The 2016 Danube Competition in Mathematics, October 29$^{\text{th}}$

1. Let ABC be a triangle, D the foot of the altitude from A and M the midpoint of the side BC. Let S be a point on the closed segment DM and let P, Q the projections of S on the lines AB and AC respectively. Prove that the length of the segment PQ does not exceed one quarter the perimeter of the triangle ABC.

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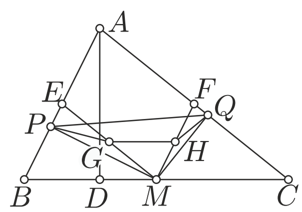

Solution. From the cyclic quadrilateral $APSQ$, whose circumcircle has diameter AS, $PQ = AS \sin \widehat{PAQ}$. So, the largest value of PQ is obtained when AS is largest, that is when $S = M$.

In the case $S = M$, denote E, F the midpoints of the segments AB, respectively AC and G, H the midpoints of the segments ME, respectively MC. Then

$$PQ \leq PG + GH + HQ = \frac{1}{2}(EM + EF + FM) = \frac{1}{4}P_{\triangle ABC},$$

as desired. Equality occurs if and only if ABC is equilateral.

2. A bank has a set S of codes for its customers, in the form of sequences of 0 and 1, each sequence being of length n. Two codes are called *close* if they are different at exactly one position. It is known that each code from S has exactly k close codes in S.

a) Show that S has an even number of elements.

b) Show that S contains at least 2^k codes.

Solution. Start by noticing that we can suppose that S contains the nil code $(0, 0, \ldots, 0)$: otherwise take a code $x \in S$ and replace S with the set $S' = x + S = \{x + y \mid y \in S\}$, where addition is taken mod 2. Then S' and S have the same cardinal and S' fulfills the same condition as S.

In the sequel we will denote $w(x)$ the number of non-nil components of the code x and $S_i = \{x \in S \mid w(x) = i\}$.

a) Consider the bipartite graph $G = A \cup B$, where the vertices are the codes, $A = \{x \in S \mid w(x) = \text{even}\}$, $B = \{x \in S \mid w(x) = \text{odd}\}$ and an edge between two vertices exists if and only if the corresponding codes are close. Count the edges of this graph: from A emerge $k \cdot |A|$ edges and from B emerge $k \cdot |B|$ edges. But $k \cdot |A| = k \cdot |B|$, hence A and B have the same number of vertices, whence the conclusion.

b) Notice first that $|S_0| = 1$. Take now $x \in S_i$. The set S_{i-1} has at most i codes close to x and the set S_{i+1} has at least $k - i$ codes close to x. On the other hand, each code from S_{i+1} has at most $i + 1$ close codes in the set S_i.

Consider now the (bipartite) subgraph $S_i \cup S_{i+1}$. We count the number N of its edges twice: first we count the edges emerging from S_i to find that $N \geq (k - i)|S_i|$, then we count the edges emerging from S_{i+1} to find that $N \leq (i + 1)|S_{i+1}|$. So, $(i + 1)|S_{i+1}| \geq (k - i)|S_i|$.

The last inequality yields inductively to $|S_i| \geq C_k^i$ for every $0 \leq i \leq k$, whence $|S| \geq 2^k$.

3. Let $n > 1$ be an integer and $a_1, a_2, \ldots, a_n$ be positive integers with sum 1.
a) Show that there exists a constant $c \geq 1/2$ so that

$$\sum_{k=1}^{n} \frac{a_k}{1+(a_0+\cdots+a_{k-1})^2} \geq c,$$

where $a_0 = 0$.
b) Show that 'the best' value of c is at least $\pi/4$.

Solution. For the first part, notice that

$$\sum_{k=1}^{n} \frac{a_k}{1+(a_0+\cdots+a_{k-1})^2} \geq \sum_{k=1}^{n} \frac{a_k}{(1+a_0+a_1+\cdots+a_{k-1})^2}.$$

Since

$$\begin{aligned}\frac{a_k}{(1+a_0+a_1+\cdots+a_{k-1})^2} &\geq \frac{a_k}{(1+a_0+a_1+\cdots+a_{k-1})(1+a_0+a_1+\cdots+a_k)}\\ &= \frac{1}{1+a_0+a_1+\cdots+a_{k-1}} - \frac{1}{1+a_0+a_1\cdots+a_k},\end{aligned}$$

the relation follows by summing.

For the second part we denote $x_k = a_0 + a_1 + \cdots + a_k$ and use the inequality

$$\frac{x_{k+1}-x_k}{1+x_k^2} \geq \arctan x_{k+1} - \arctan x_k,$$

valid for $1 \geq x_{k+1} \geq x_k \geq 0$. This is equivalent to $\tan \frac{x_{k+1}-x_k}{1+x_k^2} \geq \frac{x_{k+1}-x_k}{1+x_k x_{k+1}}$, and results from $\tan x \geq x$ for $x \in (0, \frac{\pi}{2})$.

Remark. Taking $a_k = 1/n$, $k = 1, 2, \ldots, n$ and $n \to \infty$, a limit argument shows that the value $c = \pi/4$ cannot be improved.

4. Prove that there exist only finitely many positive integers n such that

$$\left(\frac{n}{1}+1\right)\left(\frac{n}{2}+2\right)\left(\frac{n}{3}+3\right)\ldots\left(\frac{n}{n}+n\right)$$

is an integer.

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Solution. Assume that n is sufficiently large. We claim that there exists a prime number $p \leq n$ such that p does not divide $k^2 + n$, for any integer k.

Lemma. There exist coprime positive integers a, b of different parities such that

$$3n \leq (a^2+b^2)^2 \leq 4n.$$

Proof. We may take $a = 1$ and b even maximal such that $a^2 + b^2 \leq 2\sqrt{n}$, etc. □

Note that $N = 4n - (a^2+b^2)^2 \equiv 3 (\mathrm{mod} 4)$, so we can find a prime factor p of N which is of the form $4k+3$, since $N \geq 0$. We claim that p satisfies the desired property.

Recall that for a prime number $p \equiv 3(\mathrm{mod} 4)$, if p divides $x^2 + y^2$, then p divides both x and y. Assume that there was some k such that p divides $k^2 + n$. Then

$$p | (2k)^2 + (a^2+b^2)^2,$$

hence $p | a^2 + b^2$, since $p \equiv 3(\mathrm{mod} 4)$. Then, $p|a$ and $p|b$, contradicting $\gcd(a, b) = 1$.