Călăraşi 2014

Problema 1. Două cercuri secante $\mathcal{C}_1$, $\mathcal{C}_2$ au punctele comune A şi A'. Tangenta în A la $\mathcal{C}_1$ taie $\mathcal{C}_2$ în B, tangenta în A la $\mathcal{C}_2$ taie $\mathcal{C}_1$ în C, iar dreapta BC taie din nou $\mathcal{C}_1$ şi $\mathcal{C}_2$ în D_1, respectiv D_2. Se consideră punctele $E_1 \in (AD_1)$ şi $E_2 \in (AD_2)$, astfel încât $AE_1 = AE_2$. Dreptele BE_1 şi AC se intersectează în punctul M, dreptele CE_2 şi AB se intersectează în punctul N, iar dreptele MN şi BC se intersectează în punctul P. Arătaţi că PA este tangentă la cercul circumscris triunghiului ABC.

Problema 2. Fie S o mulţime de numere naturale nenule, astfel încât $\lfloor\sqrt{x}\rfloor = \lfloor\sqrt{y}\rfloor$, oricare ar fi elementele x şi y ale lui S. Arătaţi că produsele xy, unde $x, y \in S$, sunt distincte două câte două.

Problema 3. Arătaţi că, oricare ar fi numărul întreg $n \geq 2$, există o mulţime de n numere întregi compuse, coprime două câte două, care formează o progresie aritmetică.

Problema 4. Fie n un număr natural nenul şi fie Δ triunghiul cu vârfurile în punctele laticiale $(0,0)$, $(n,0)$ şi $(0,n)$. Determinaţi cardinalul maxim al unei mulţimi S de puncte laticiale situate în interiorul sau pe bordul lui Δ, astfel încât segmentul determinat de oricare două puncte distincte din S să nu fie paralel cu niciuna dintre laturile lui Δ.

Călăraşi 2014 — Solutions

Problem 1. Two circles γ_1 and γ_2 cross one another at two points; let A be one of these points. The tangent to γ_1 at A meets again γ_2 at B, the tangent to γ_2 at A meets again γ_1 at C, and the line BC meets again γ_1 and γ_2 at D_1 and D_2, respectively. Let E_1 and E_2 be interior points of the segments AD_1 and AD_2, respectively, such that $AE_1 = AE_2$. The lines BE_1 and AC meet at M, the lines CE_2 and AB meet at N, and the lines MN and BC meet at P. Show that the line PA is tangent to the circle ABC.

Solution. We shall prove that $PA^2 = PB \cdot PC$. By Stewart's relation, $PA^2 \cdot BC \mp AB^2 \cdot PC \pm AC^2 \cdot PB = PB \cdot PC \cdot BC$, this amounts to showing $PB \cdot AC^2 = PC \cdot AB^2$.

To begin, apply Menelaus' theorem to triangles ABD_2, ACD_1, ABC and transversals NE_2C, ME_1B, MNP, respectively, to write

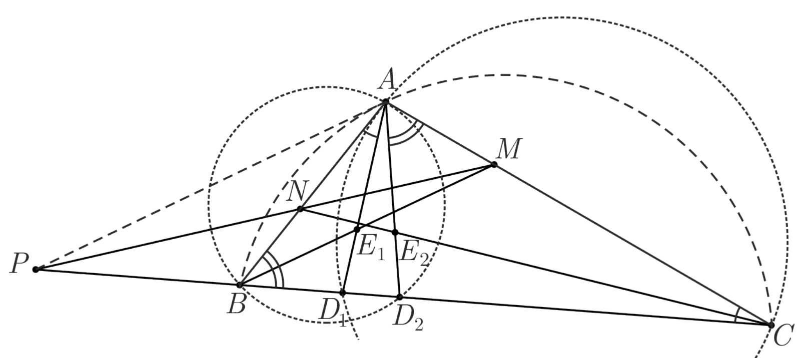
$$\frac{NB}{NA} \cdot \frac{CD_2}{CB} \cdot \frac{E_2A}{E_2D_2} = 1, \quad \frac{MA}{MC} \cdot \frac{E_1D_1}{E_1A} \cdot \frac{BC}{BD_1} = 1, \quad \frac{MC}{MA} \cdot \frac{NA}{NB} \cdot \frac{PB}{PC} = 1,$$

so, multiplying the three,

$$\frac{E_1D_1}{E_2D_2} \cdot \frac{CD_2}{BD_1} \cdot \frac{PB}{PC} = 1, \qquad (*)$$

on account of $AE_1 = AE_2$. Since $\angle AD_1B = \angle BAC = \angle AD_2C$, it follows that $AD_1 = AD_2$, so $E_1D_1 = E_2D_2$, with reference again to $AE_1 = AE_2$. Consequently, $PB/PC = BD_1/CD_2$, by $(*)$.

Finally, similarity of the triangles ABC and D_1BA yields $BD_1 = AB^2/BC$. Similarly, $CD_2 = AC^2/BC$, so $PB \cdot AC^2 = PC \cdot AB^2$, by the preceding. This ends the proof.



Problem 2. Let S be a set of positive integers such that $\lfloor\sqrt{x}\rfloor = \lfloor\sqrt{y}\rfloor$ for all $x, y \in S$. Show that the products xy, where $x, y \in S$, are pairwise distinct.

Solution. We first show that if x_1, x_2, x_3, x_4 are (not necessarily distinct) members of S such that $x_1x_2 \le x_3x_4$, then $x_1 + x_2 \le x_3 + x_4$.

Suppose, if possible, that $x_1 + x_2 > x_3 + x_4$. Let $n = \lfloor\sqrt{x}\rfloor$, $x \in S$, and write $x_k = n^2 + w_k$, where the w_k are non-negative integers less than $2n+1$, to deduce that $w_1 + w_2 - w_3 - w_4 \ge 1$. The condition $x_1x_2 \le x_3x_4$ yields $(w_1 + w_2 - w_3 - w_4)n^2 \le w_3w_4 - w_1w_2$, so $w_3 > 0$ and

$$\begin{aligned} n^2 &\le (w_1 + w_2 - w_3 - w_4)n^2 \le w_3w_4 - w_1w_2 < w_3(w_1 + w_2 - w_3) - w_1w_2 \\ &= (w_1 - w_3)(w_3 - w_2) \le ((w_1 - w_3) + (w_3 - w_2))^2/4 = (w_1 - w_2)^2/4 \le n^2, \end{aligned}$$

which is a contradiction.

Thus, if x_1, x_2, x_3, x_4 are members of S such that $x_1x_2 = x_3x_4$, then $x_1 + x_2 = x_3 + x_4$, so $x_1^2 + x_3x_4 = x_1(x_1 + x_2) = x_1(x_3 + x_4)$, i.e., $(x_1 - x_3)(x_1 - x_4) = 0$ whence $x_1 = x_3$ or $x_1 = x_4$. The conclusion now follows at once.

Remark. The result is sharp, in the sense that the conclusion may fail if the square roots of the members of S do not all have the same integral part. This is the case if, for instance, n^2, $n^2 + n$ and $(n+1)^2$ are all members of S, since $n^2(n+1)^2 = (n^2+n)(n^2+n)$.

Problem 3. Given any integer $n \geq 2$, show that there exists a set of n pairwise coprime composite integers in arithmetic progression.

Solution. Fix a prime $p > n$ and an integer $N \geq p + (n-1)n!$ and consider the arithmetic progression of length n consisting of the numbers $N! + p + kn!$, $k = 0, 1, \ldots, n-1$.

Suppose, if possible, that q is a prime factor of two of these numbers. Then q divides their difference which is of the form $kn!$, for some positive integer $k < n$. It follows that q does not exceed n, so $n!$ and $N!$ are both divisible by q, and consequently so is p — a contradiction.

Problem 4. Let n be a positive integer and let Δ be the closed triangular domain with vertices at the lattice points $(0,0)$, $(n,0)$ and $(0,n)$. Determine the maximal cardinality a set S of lattice points in Δ may have, if the line through every pair of distinct points in S is parallel to no side of Δ.

Solution. The required maximum is $\lfloor 2n/3 \rfloor + 1$ and is achieved, for instance, for

$$S = \{(2k, \lfloor n/3 \rfloor - k) \colon k = 0, \ldots, \lfloor n/3 \rfloor\} \cup \{(2k+1, 2\lfloor n/3 \rfloor - k) \colon k = 0, \ldots, \lfloor n/3 \rfloor - 1\},$$

if $n \equiv 0$ or $n \equiv 1$ modulo 3, and

$$S = \{(2k, \lfloor n/3 \rfloor - k) \colon k = 0, \ldots, \lfloor n/3 \rfloor\} \cup \{(2k+1, 2\lfloor n/3 \rfloor - k + 1) \colon k = 0, \ldots, \lfloor n/3 \rfloor\},$$

if $n \equiv 2$ modulo 3.

If (x, y) is a point in Δ, and $z = z(x, y)$ is the distance from (x, y) to the side through $(n, 0)$ and $(0, n)$, then

$$x + y + z\sqrt{2} = n; \tag{1}$$

and if, in addition, (x, y) is a lattice point, then x, y and $z\sqrt{2}$ are all non-negative integers (not exceeding n).

Now, let S be a set of lattice points in Δ satisfying the condition in the statement, and sum (1) over all points (x, y) in S to get

$$\sum_{(x,y) \in S} x + \sum_{(x,y) \in S} y + \sum_{(x,y) \in S} z\sqrt{2} = n|S|. \tag{2}$$

As (x, y) runs through S, each of the three coordinates x, y and $z\sqrt{2}$ runs through $|S|$ non-negative disitnct integers, so each of the three sums in (2) is greater than or equal to $0 + 1 + \cdots + (|S| - 1) = |S|(|S| - 1)/2$. Consequently, $3|S|(|S| - 1)/2 \leq n|S|$, so $|S| \leq 2n/3 + 1$ and the conclusion follows.